Physics: Mechanics
Subject code: BSC-PHY-104G

CE

IIst Semester

Unit 3: Rigid Body Mechanics

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Rigid body

- **Rigid body: a system of mass points subject to the holonomic constraints that the distances between all pairs of points remain constant throughout the motion**

- **If there are N free particles, there are 3N degrees of freedom**

- **For a rigid body, the number of degrees of freedom is reduced by the constraints expressed in the form:**

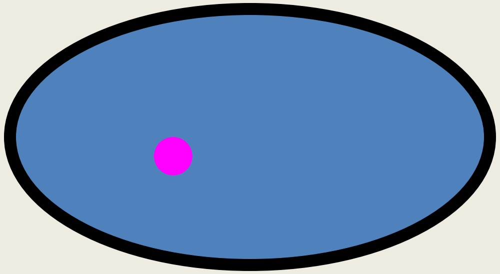
$$r_{ij} = c_{ij}$$

- **How many independent coordinates does a rigid body have?**

The independent coordinates of a rigid body

• Rigid body has to be described by its orientation and location

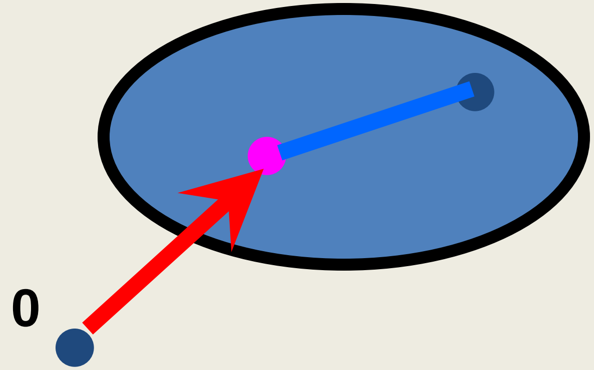
• Position of the rigid body is determined by the position of any one point of the body, and the orientation is determined by the relative position of all other points of the body relative to that point



The independent coordinates of a rigid body

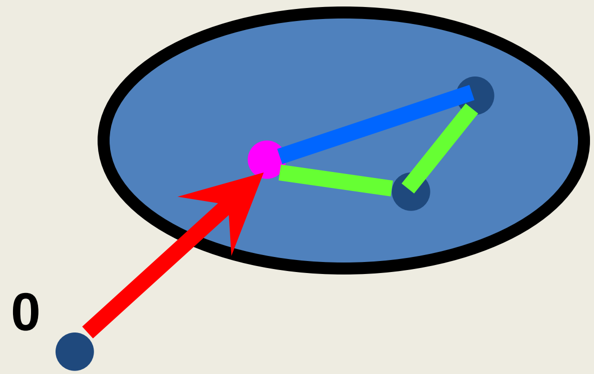
• **Position of one point of the body requires the specification of 3 independent coordinates**

• **The position of a second point lies at a fixed distance from the first point, so it can be specified by 2 independent angular coordinates**



• **The position of any other third point is determined by only 1 coordinate, since its distance from the first and second points is fixed**

• **Thus, the total number of independent coordinates necessary do completely describe the position and orientation of a rigid body is 6**



Orientation of a rigid body

- The position of a rigid body can be described by three independent coordinates,

- Therefore, the orientation of a rigid body can be described by the remaining three independent coordinates

- There are many ways to define the three orientation coordinates

- One common ways is via the definition of direction cosines

Direction cosines

- **Direction cosines specify the orientation of one Cartesian set of axes relative to another set with common origin**

$$\hat{i}' = \hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13}$$

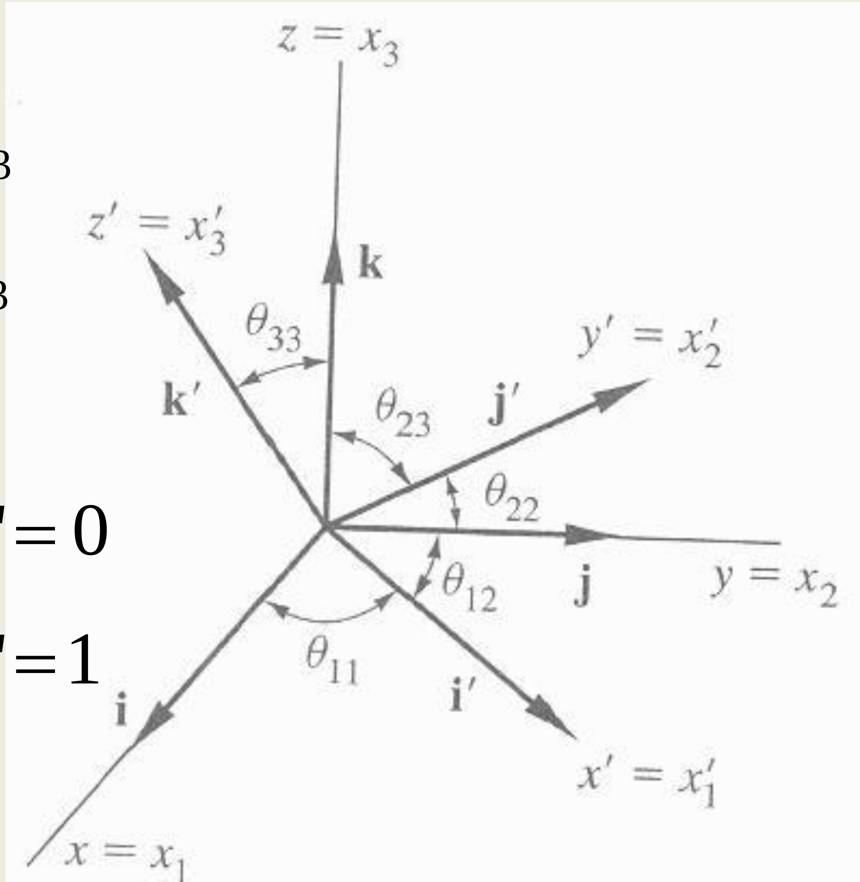
$$\hat{j}' = \hat{i}\cos\theta_{21} + \hat{j}\cos\theta_{22} + \hat{k}\cos\theta_{23}$$

$$\hat{k}' = \hat{i}\cos\theta_{31} + \hat{j}\cos\theta_{32} + \hat{k}\cos\theta_{33}$$

- **Orthogonality conditions:**

$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = \hat{i}'\cdot\hat{j}' = \hat{j}'\cdot\hat{k}' = \hat{k}'\cdot\hat{i}' = 0$$

$$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = \hat{i}'\cdot\hat{i}' = \hat{j}'\cdot\hat{j}' = \hat{k}'\cdot\hat{k}' = 1$$



Orthogonality conditions

$$\hat{i}'\cdot\hat{i}' =$$

$$= (\hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13})\cdot(\hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13})$$

$$= \cos^2\theta_{11} + \cos^2\theta_{12} + \cos^2\theta_{13} \;\; = 1$$

$$\hat{i}'\cdot\hat{j}' =$$

$$= (\hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13})\cdot(\hat{i}\cos\theta_{21} + \hat{j}\cos\theta_{22} + \hat{k}\cos\theta_{23})$$

$$= \cos\theta_{11}\cos\theta_{21} + \cos\theta_{12}\cos\theta_{22} + \cos\theta_{13}\cos\theta_{23} = 0$$

- **Performing similar operations for the remaining 4 pairs we obtain orthogonality conditions in a compact form:**

$$\boxed{\sum_{l=1}^{3}\cos\theta_{li}\cos\theta_{lk} = \delta_{ik}}$$

Orthogonal transformations

- **For an arbitrary vector** $\vec{G} = \hat{i}G_1 + \hat{j}G_2 + \hat{k}G_3$

- **We can find components in the primed set of axes as follows:**

$$G_1' = \hat{i}'\cdot\vec{G} = \hat{i}'\cdot\hat{i}G_1 + \hat{i}'\cdot\hat{j}G_2 + \hat{i}'\cdot\hat{k}G_3$$

$$= (\hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13})\cdot\hat{i}G_1$$

$$+ (\hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13})\cdot\hat{j}G_2$$

$$+ (\hat{i}\cos\theta_{11} + \hat{j}\cos\theta_{12} + \hat{k}\cos\theta_{13})\cdot\hat{k}G_3$$

$$= \cos\theta_{11}G_1 + \cos\theta_{12}G_2 + \cos\theta_{13}G_3$$

- **Similarly**

$$G_2' = \cos\theta_{21}G_1 + \cos\theta_{22}G_2 + \cos\theta_{23}G_3$$

$$G_3' = \cos\theta_{31}G_1 + \cos\theta_{32}G_2 + \cos\theta_{33}G_3$$

Orthogonal transformations

- Therefore, orthogonal transformations are defined as:

$$\boxed{G_i' = \sum_{j=1}^{3} a_{ij} G_j; \qquad a_{ij} \equiv \cos\theta_{ij}}$$

- Orthogonal transformations can be expressed as a matrix relationship with a transformation matrix A

$$\mathbf{G}' = \mathbf{A}\mathbf{G}$$

- With orthogonality conditions imposed on the transformation matrix A

$$\sum_{l=1}^{3} a_{li} a_{lk} = \delta_{ik}$$

Properties of the transformation matrix

- **Introducing a matrix** **inverse** **to the transformation matrix**

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{1} \qquad \sum_{l=1}^{3} a_{kl}\bar{a}_{li} = \delta_{ki}$$

- **Let us consider a matrix element** $a_{ij} = \sum_{k=1}^{3} a_{kj}\delta_{ki}$

$$= \sum_{k=1}^{3}\left(a_{kj}\left(\sum_{l=1}^{3} a_{kl}\bar{a}_{li}\right)\right) = \sum_{k=1}^{3}\sum_{l=1}^{3} a_{kj}a_{kl}\bar{a}_{li} = \sum_{l=1}^{3}\left(\bar{a}_{li}\left(\sum_{k=1}^{3} a_{kj}a_{kl}\right)\right)$$

$$= \sum_{l=1}^{3}\bar{a}_{li}\delta_{jl} = \bar{a}_{ji} = a_{ij}$$

$$\boxed{\mathbf{A}^{-1} = \tilde{\mathbf{A}}}$$

- **Orthogonality conditions** $\sum_{k=1}^{3} a_{kj}a_{kl} = \delta_{jl}$

Properties of the transformation matrix

$$\tilde{\mathbf{A}} = \mathbf{A}^{-1} \qquad \mathbf{A}\tilde{\mathbf{A}} = \mathbf{A}\mathbf{A}^{-1} \qquad \boxed{\mathbf{A}\tilde{\mathbf{A}} = \mathbf{1}}$$

- **Calculating the determinants**

$$\left|\mathbf{A}\tilde{\mathbf{A}}\right| = |\mathbf{A}|\left|\tilde{\mathbf{A}}\right| = |\mathbf{A}||\mathbf{A}| = |\mathbf{A}|^2 = |\mathbf{1}| = 1$$

$$\therefore \boxed{|\mathbf{A}| = \pm 1}$$

- **The case of a negative determinant corresponds to a complete inversion of coordinate axes and is not physical (a.k.a. improper)**

Properties of the transformation matrix

- **In a general case, there are 9 non-vanishing elements in the transformation matrix**

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- **In a general case, there are 6 independent equations in the orthogonality conditions**

$$\sum_{l=1}^{3} \cos\theta_{li} \cos\theta_{lk} = \delta_{ik}$$

$$\hat{i}'\cdot\hat{j}' = \hat{j}'\cdot\hat{k}' = \hat{k}'\cdot\hat{i}' = 0$$

$$\hat{i}'\cdot\hat{i}' = \hat{j}'\cdot\hat{j}' = \hat{k}'\cdot\hat{k}' = 1$$

- **Therefore, there are 3 independent coordinates that describe the orientation of the rigid body**

Example: rotation in a plane

- **Let's consider a 2D rotation of a position vector *r***

- **The z component of the vector is not affected, therefore the transformation matrix should look like**

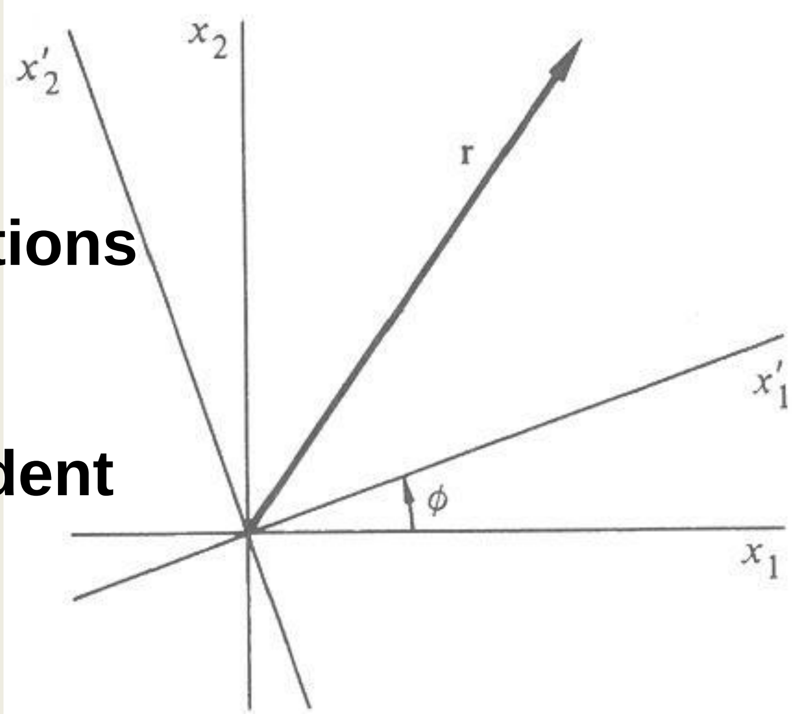
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **With the orthogonality conditions**

$$\sum_{k=1}^{2} a_{kj} a_{kl} = \delta_{jl} \qquad\qquad j,l = 1,2$$

- **The total number of independent coordinates is**

4 – 3 = 1



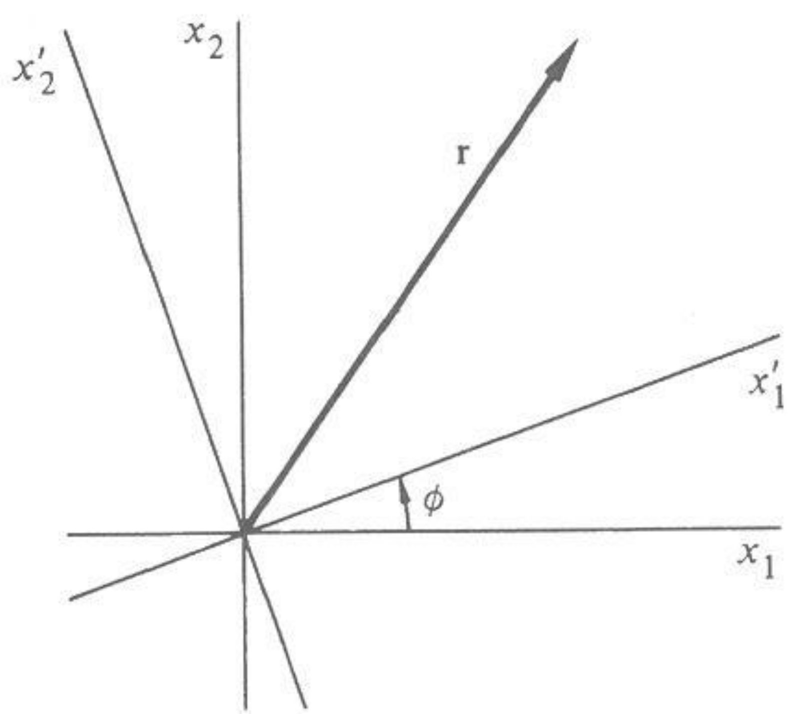
Example: rotation in a plane

- **The most natural choice for the independent coordinate would be the angle of rotation, so that**

$$x_1' = x_1 \cos\phi + x_2 \sin\phi$$
$$x_2' = -x_1 \sin\phi + x_2 \cos\phi$$
$$x_3' = x_3$$

- **The transformation matrix**

$$\mathbf{A} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



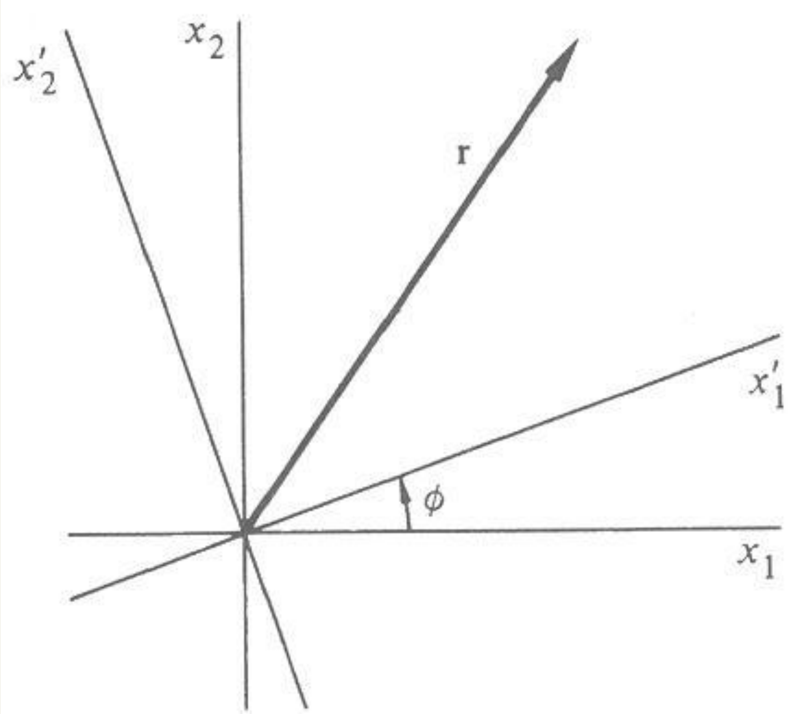
Example: rotation in a plane

- **The three orthogonality conditions**

$$a_{11}a_{11} + a_{21}a_{21} = 1$$
$$a_{12}a_{12} + a_{22}a_{22} = 1$$
$$a_{11}a_{12} + a_{21}a_{22} = 0$$

- **They are rewritten as**

$$\cos^2\phi + \sin^2\phi = 1$$
$$\sin^2\phi + \cos^2\phi = 1$$
$$\cos\phi\sin\phi - \sin\phi\cos\phi = 0$$

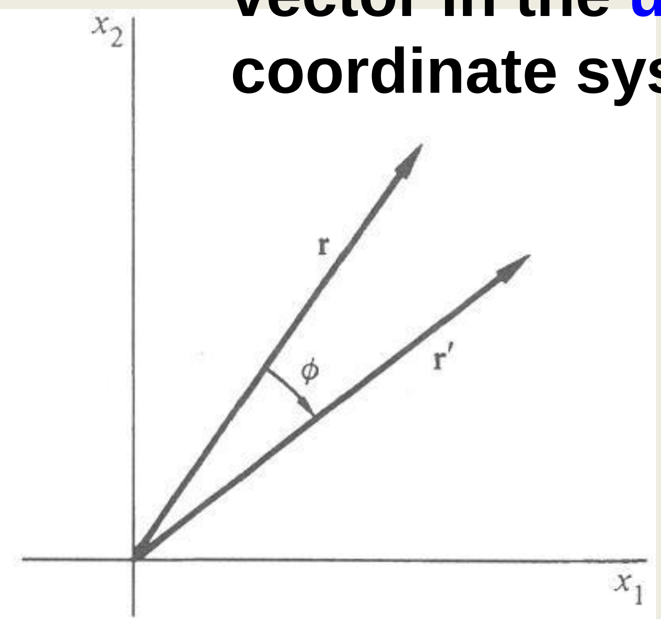


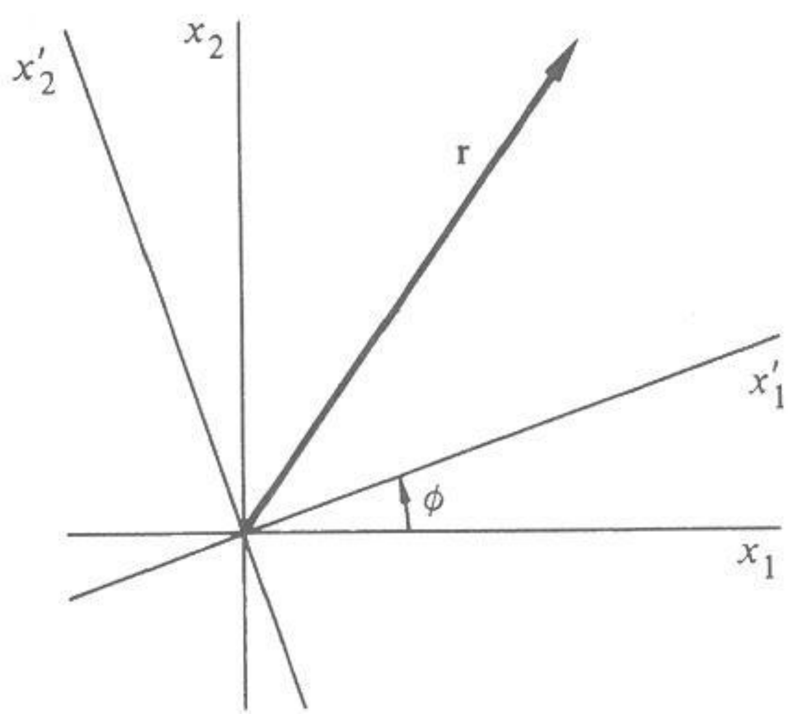
Example: rotation in a plane

- **The 2D transformation matrix**

$$\mathbf{A} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **It describes a CCW rotation of the coordinate axes**
- **Alternatively, it can describe a CW rotation of the same vector in the unchanged coordinate system**





The Euler angles

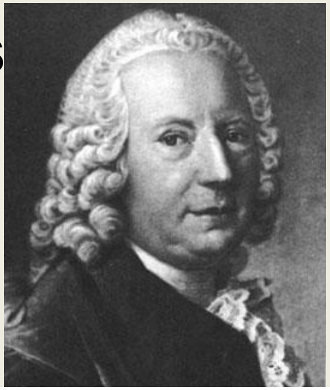
• In order to describe the motion of rigid bodies in the canonical formulation of mechanics, it is necessary to seek three independent parameters that specify the orientation of a rigid body

• The most common and useful set of such parameters are the Euler angles

• The Euler angles correspond to an orthogonal transformation via three successive rotations performed in a specific sequence

• The Euler transformation matrix is proper

$$|\mathbf{A}| = 1$$

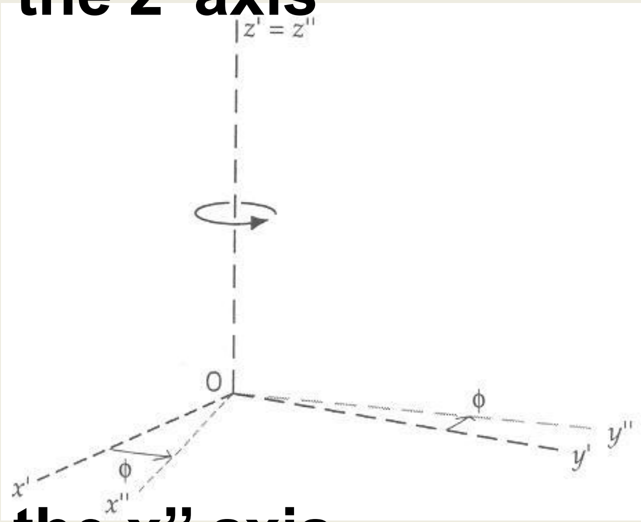


Leonhard Euler
(1707 – 1783)

The Euler angles

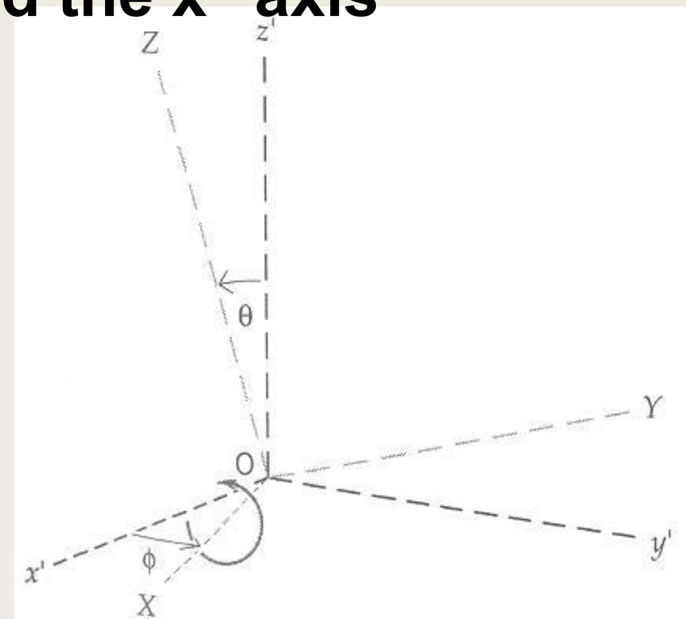
- First, we rotate the system around the z' axis

$$\mathbf{x}'' = \mathbf{D}\mathbf{x}' = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$



- Then we rotate the system around the x'' axis

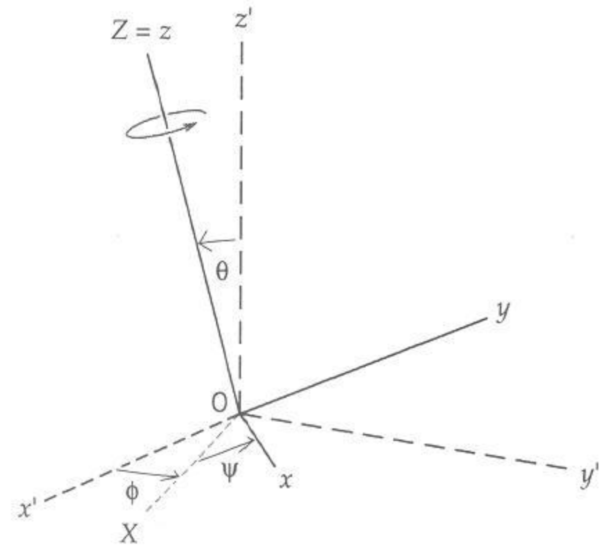
$$\mathbf{X} = \mathbf{C}\mathbf{x}'' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}$$



The Euler angles

- **Finally, we rotate the system around the Z axis**

$$\mathbf{x} = \mathbf{B}\mathbf{X} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$



- **The complete transformation can be expressed as a product of the successive matrices**

$$\mathbf{x} = \mathbf{B}\mathbf{X} = \mathbf{B}\mathbf{C}\mathbf{x}'' = \mathbf{B}\mathbf{C}\mathbf{D}\mathbf{x}' \equiv \mathbf{A}\mathbf{x}'$$

$$\boxed{\mathbf{x} = \mathbf{A}\mathbf{x}'}$$

The Euler angles

- **The explicit form of the resultant transformation matrix A is**

$$\mathbf{A} = \mathbf{BCD} =$$

$$= \begin{bmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

- **The described sequence is known as the x-convention**

- **Overall, there are 12 different possible conventions in defining the Euler angles**

Euler theorem

- **Euler theorem: the general displacement of a rigid body with one point fixed is a rotation about some axis**

- **If the fixed point is taken as the origin, then the displacement of the rigid body involves no translation; only the change in orientation**

- **If such a rotation could be found, then the axis of rotation would be unaffected by this transformation**

- **Thus, any vector lying along the axis of rotation must have the same components before and after the orthogonal transformation:**

$$\mathbf{R}' = \mathbf{A}\mathbf{R} = \mathbf{R}$$

Euler theorem

$$\mathbf{AR} = \mathbf{R} \qquad \mathbf{AR} = \mathbf{1R} \qquad (\mathbf{A} - \mathbf{1})\mathbf{R} = 0$$

- This formulation of the Euler theorem is equivalent to an eigenvalue problem $(\mathbf{A} - \lambda\mathbf{1})\mathbf{R} = 0$

- With one of the eigenvalues $\lambda = 1$

- So we have to show that the orthogonal transformation matrix has at least one eignevalue $\lambda = 1$

- The secular equation of an eigenvalue problem is

$$\left|\mathbf{A} - \lambda\mathbf{1}\right| = 0$$

- It can be rewritten for the case of $\lambda = 1$

$$\left|\mathbf{A} - \mathbf{1}\right| = 0$$

Euler theorem

- **Recall the orthogonality condition:** $|\mathbf{A}| = 1 \quad \mathbf{A}\tilde{\mathbf{A}} = \mathbf{1}$

$$\mathbf{A}\tilde{\mathbf{A}} - \tilde{\mathbf{A}} = \mathbf{1} - \tilde{\mathbf{A}} \qquad (\mathbf{A}-\mathbf{1})\tilde{\mathbf{A}} = \mathbf{1} - \tilde{\mathbf{A}} \qquad \left|(\mathbf{A}-\mathbf{1})\tilde{\mathbf{A}}\right| = \left|\mathbf{1} - \tilde{\mathbf{A}}\right|$$

$$|\mathbf{A}-\mathbf{1}|\left|\tilde{\mathbf{A}}\right| = \left|\mathbf{1} - \tilde{\mathbf{A}}\right| \qquad |\mathbf{A}-\mathbf{1}||\mathbf{A}| = \left|\mathbf{1} - \tilde{\mathbf{A}}\right| \qquad |\mathbf{A}-\mathbf{1}| = \left|\mathbf{1} - \tilde{\mathbf{A}}\right|$$

$$|\mathbf{A}-\mathbf{1}| = \left|\widetilde{\mathbf{1} - \mathbf{A}}\right| \qquad |\mathbf{A}-\mathbf{1}| = |\mathbf{1} - \mathbf{A}| \qquad |\mathbf{A}-\mathbf{1}| = |-(\mathbf{A}-\mathbf{1})|$$

$$|\mathbf{A}-\mathbf{1}| = (-1)^n |\mathbf{A}-\mathbf{1}|$$

- ***n*** **is the dimension of the square matrix**

- **For 3D case:** $|\mathbf{A}-\mathbf{1}| = (-1)^3 |\mathbf{A}-\mathbf{1}| \qquad |\mathbf{A}-\mathbf{1}| = -|\mathbf{A}-\mathbf{1}|$

- **It can be true only if** $\boxed{|\mathbf{A}-\mathbf{1}| = 0}$ $\qquad Q.E.D.$

Euler theorem

Michel Chasles
(1793–1880)

- **For 2D case (rotation in a plane) *n = 2*:**

$$|\mathbf{A}-\mathbf{1}| = (-1)^n |\mathbf{A}-\mathbf{1}| \qquad |\mathbf{A}-\mathbf{1}| = |\mathbf{A}-\mathbf{1}|$$

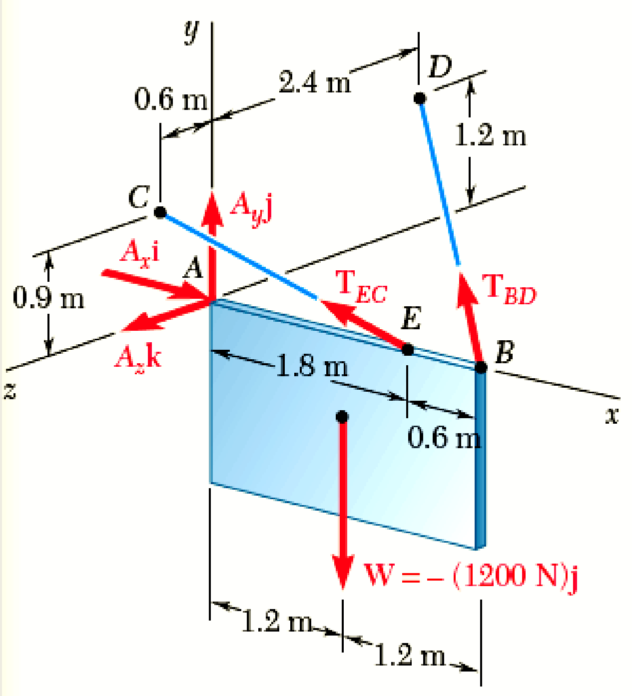
- **Euler theorem does not hold for all orthogonal transformation matrices in 2D: there is no vector in the plane of rotation that is left unaltered – only a point**

- **To find the direction of the rotation axis one has to solve the system of equations for three components of vector *R*:**

$$(\mathbf{A}-\mathbf{1})\mathbf{R} = 0$$

- **Removing the constraint, we obtain Chasles' theorem: the most general displacement of a rigid body is a translation plus a rotation**

Sample Problem 4.8



- Create a free-body diagram for the sign.

  Since there are only 5 unknowns, the sign is partially constrain. It is free to rotate about the x axis. It is, however, in equilibrium for the given loading.

$$\vec{T}_{BD} = T_{BD}\frac{\vec{r}_D - \vec{r}_B}{\left|\vec{r}_D - \vec{r}_B\right|}$$

$$= T_{BD}\frac{-2.4\vec{i} + 1.2\vec{j} - 2.4\vec{k}}{3.6}$$

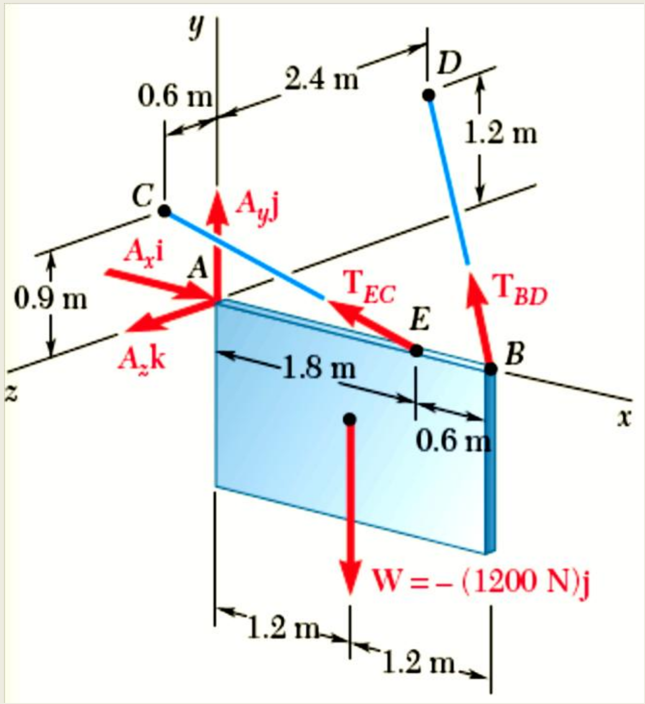
$$= T_{BD}\left(-\tfrac{2}{3}\vec{i} + \tfrac{1}{3}\vec{j} - \tfrac{2}{3}\vec{k}\right)$$

$$\vec{T}_{EC} = T_{EC}\frac{\vec{r}_C - \vec{r}_E}{\left|\vec{r}_C - \vec{r}_E\right|}$$

$$= T_{EC}\frac{-6\vec{i} + 3\vec{j} + 2\vec{k}}{7}$$

$$= T_{EC}\left(-\tfrac{6}{7}\vec{i} + \tfrac{3}{7}\vec{j} + \tfrac{2}{7}\vec{k}\right)$$

Sample Problem 4.8



$$\sum \vec{F} = \vec{A} + \vec{T}_{BD} + \vec{T}_{EC} - (1200\text{ N})\vec{j} = 0$$

$$\vec{i}: \quad A_x - \tfrac{2}{3}T_{BD} - \tfrac{6}{7}T_{EC} = 0$$

$$\vec{j}: \quad A_y + \tfrac{1}{3}T_{BD} + \tfrac{3}{7}T_{EC} - 1200\text{ N} = 0$$

$$\vec{k}: \quad A_z - \tfrac{2}{3}T_{BD} + \tfrac{2}{7}T_{EC} = 0$$

$$\sum \vec{M}_A = \vec{r}_B \times \vec{T}_{BD} + \vec{r}_E \times \vec{T}_{EC} + (1.2\text{ m})\vec{i} \times (-1200\text{ N})\vec{j} = 0$$

$$\vec{j}: \quad 1.6T_{BD} - 0.514T_{EC} = 0$$

$$\vec{k}: \quad 0.8T_{BD} + 0.771T_{EC} - 1440\text{ N.m} = 0$$

- Apply the conditions for static equilibrium to develop equations for the unknown reactions.

Solve the 5 equations for the 5 unknowns,

$$\boxed{\begin{gathered} T_{BD} = 451\text{ N} \quad T_{EC} = 1402\text{ N} \\ \vec{A} = (1502\text{ N})\vec{i} + (419\text{ N})\vec{j} - (100.1\text{ N})\vec{k} \end{gathered}}$$

Future Scope and relevance to industry

- https://www.researchgate.net/publication/281457879_EXTENSION_OF_EULER'S_THEOREM_ON_HOMOGENEOUS_FUNCTION_TO_HIGHER_DERIVATIVES
- https://journals.plos.org/plosone/article?id=10.1371/journal.pone.0026308
- https://ntrs.nasa.gov/search.jsp?R=19660027747

NPTEL/other online link

- https://nptel.ac.in/courses/115105098/34
- https://nptel.ac.in/courses/105106116/32
- https://nptel.ac.in/courses/111104095/15